HIGH SCHOOL MATHEMATICS CONTESTS

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Contest Number 4 — *Any calculator is always allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. — **February 7, 1995**

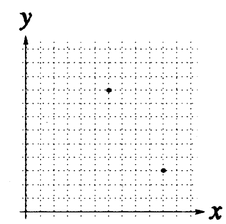
Name ______________________ **Teacher** ___________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes — *Answer Column*

4-1. If the lengths of two sides of a right triangle are 3 and 4, what is the least possible length of the third side? — 4-1.

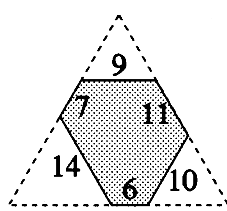
4-2. What is the integer n for which $5^n + 5^n + 5^n + 5^n + 5^n = 5^{25}$? — 4-2.

4-3. If (6,9) and (10,3) are the coordinates of two opposite vertices of a square, what are the coordinates of the other two vertices? — 4-3.



4-4. If i represents the imaginary unit, what is the ordered pair of real numbers (a,b) for which $(1 + i)^{13} = a + bi$? — 4-4.

4-5. The diagram shows that an equiangular hexagon with side-lengths 6, 7, 9, 10, 11, and 14 can be inscribed in an equilateral triangle with side-length 30. This same equiangular hexagon can also be inscribed in an equilateral triangle with side-length $n \neq 30$. What is this value of n? — 4-5.



4-6. In a certain sequence, the first number is 1995. The second number equals the first number divided by 1 more than the first number. The third number equals the second number divided by 1 more than the second number. From then on, each number in the sequence equals the previous number divided by 1 more than the previous number. What is the 1995th number in this sequence? — 4-6.

Problem 4-1

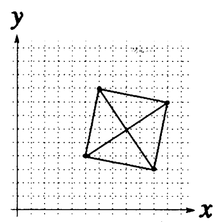
There are two possibilities: the lengths of the legs could be 3 and 4 (making the length of the hypotenuse 5), or the length of one leg could be 3 and the length of the hypotenuse could be 4, making the length of the third side $= \sqrt{4^2 - 3^2} = \boxed{\sqrt{7}}$.

Problem 4-2

If $5^n + 5^n + 5^n + 5^n + 5^n = 5 \times 5^n = 5^{n+1} = 5^{25}$, then $n + 1 = 25$ and $n = \boxed{24}$.

Problem 4-3

The given vertices lie on one of the square's diagonals. Since the slope of this diagonal is $-\frac{3}{2}$, the slope of the other diagonal is $\frac{2}{3}$. The midpoint of both diagonals is (8,6). The other vertices are at $\boxed{(5,4),\ (11,8)}$.



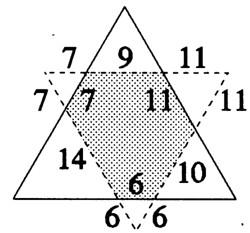
Problem 4-4

Method I: Since $(1+i)^2 = 2i$, we know $(1+i)^{13} = (2i)^6(1+i) = (64i^6)(1+i) = (-64)(1+i) = -64-64i$, so $(a,b) = \boxed{(-64,-64)}$.

Method II: By DeMoivre's Theorem, since $(1+i) = \sqrt{2}\,\text{cis}\,45°$, we know that $(1+i)^{13} = (\sqrt{2})^{13}\,\text{cis}\,585° = 64\sqrt{2}\,\text{cis}\,225° = -64 + -64i = -64 - 64i$.

Problem 4-5

As can be seen in the diagram at the right, there are two equilateral triangles that can circumscribe this hexagon. The larger triangle has a side-length of 30, and the smaller one has a side-length of $\boxed{27}$.



Problem 4-6

We have $a_{n+1} = \dfrac{a_n}{1+a_n}$. Hence, $\dfrac{1}{a_{n+1}} = \dfrac{1+a_n}{a_n}$, or $\dfrac{1}{a_{n+1}} = \dfrac{1}{a_n} + 1$. Let's use this recursion repeatedly and see what develops:

$$\frac{1}{a_1} = \frac{1}{1995}.$$
$$\frac{1}{a_2} = \frac{1}{a_1}+1 = \frac{1}{1995}+1.$$
$$\frac{1}{a_3} = \frac{1}{a_2}+1 = \left(\frac{1}{a_1}+1\right)+1 = \frac{1}{a_1}+2 = \frac{1}{1995}+2.$$
$$\frac{1}{a_4} = \frac{1}{a_3}+1 = \left(\frac{1}{a_1}+2\right)+1 = \frac{1}{a_1}+3 = \frac{1}{1995}+3.$$
$$\frac{1}{a_5} = \frac{1}{a_4}+1 = \left(\frac{1}{a_1}+3\right)+1 = \frac{1}{a_1}+4 = \frac{1}{1995}+4.$$

By induction,

$$\frac{1}{a_{1995}} = \frac{1}{1995}+1994 = \frac{1+1994\times 1995}{1995} = \frac{3978031}{1995}.$$

Taking reciprocals,

$$a_{1995} = \boxed{\frac{1995}{3978031} \text{ or } 0.0005015}.$$

[**NOTE:** Since answers must be exact or must have 4 (or more) significant digits, correctly rounded, an answer of 0.0005 must **NOT** receive credit. The answer 0.0005 has only 1 significant digit, not the 4 (or more) that are required for an approximate answer to receive credit.]