Contest Number 5  

February 12, 2019

Time Limit: 30 minutes

Name ____________________________  Teacher ____________________________  Grade Level ______  Score ______

5-1. If \( N \) is a 3-digit integer, and given that the result of reversing the digits of \( N \) is the number \( M \), what is the maximum value of \( N-M \)? [Note: Integers written with 1 or more leading zeros can also be written with every leading zero removed.]

5-2. If \( a \) and \( b \) are integers whose product is 5, what is the least possible value of \( a^b \)?

5-3. Al’s biplane can fly two different types of square advertising signs. Both signs have integer side-lengths. Their areas are \( n+1 \) and \( 2n+1 \). What is the least positive integer \( n \) for which each sign’s area is the square of an integer?

5-4. In a group of 100, each a Borg or a Corg, every Borg has 2 Corg friends and no Corg has more than 1 Borg friend. If exactly 13 Corgs have no Borg friends, then how many Corgs are in the group?

5-5. If \( 2019! \) is the product of the first 2019 positive integers, for which positive integer \( k \) is \( x = 2019! \) the only value of \( x > 0 \) that satisfies

\[
\frac{1}{\log_4 x} + \frac{1}{\log_9 x} + \frac{1}{\log_{16} x} + \cdots + \frac{1}{\log_{(2019^2)} x} = k
\]

5-6. Midpoints of two sides of a square are vertices of the shaded triangle shown. Drawing the diagonal of the square pictured splits this triangle into two parts, one of which is a trapezoid. If the area of the square is 192, what is the area of the trapezoid?
Problem 5-1
Since making N's hundreds' digit a 9 and its units' digit a 0 will maximize the difference, no matter what the middle digit is, \(N-M = 920 - 029 = 891\).

Problem 5-2
Since \(ab = 5\), and \(a\) and \(b\) are integers, \((a,b)\) must be one of \((1,5), (5,1), (-1,-5),\) or \((-5,-1)\). The least value of \(ab\) is \((-1)(-5) = 5\).

Problem 5-3
By trial and error, take each positive perfect square and subtract 1 to find every positive integer \(n\) for which \(n+1\) is a perfect square. Find the least of those integers \(n\) for which \(2n+1\) is the square of an integer. Let's try 1. That fails because \(1-1 = 0\) is not positive. Try 4: \(4-1 = 3\), but \(2(3)+1 = 7\), which is not a perfect square. We try 9, then 16, and when we finally try 25, we get \(25-1 = 24\), and \(2(24)+1 = 49\). Both \(n+1\) and \(2n+1\) are perfect squares when \(n = 24\).

Problem 5-4
If \(B\) is the number of Borgs, and \(C\) is the number of Corgs, then the number of Corgs is \(2B+13\). Therefore, \(B + (2B+13) = 100\), \(B = 29\), and \(C = 71\).

Problem 5-5
Whenever \(a > 1\) and \(b > 1\), \(\log_a b = \frac{1}{\log_b a}\). We can write \(\log_2 4 + \log_3 9 + \log_4 16 + \ldots + \log_{2019} 2019^2 = k\), or \(\log_2 (2^2 \times 3^2 \times 4^2 \times \ldots \times 2019^2) = k\). It follows that \(x^k = (2^2 \times 3^2 \times 4^2 \times \ldots \times 2019^2) = (2019!)^2\), so \(k = 2\).

Problem 5-6
Method I: The square's side is \(8\sqrt{3}\), so its diagonal is \(8\sqrt{6} = d\). The trapezoid's height is \(\frac{d}{4}\). Its longer base is \(\frac{9}{4}\). The shaded \(\triangle\)'s legs trisect the diagonal (to prove this, use similar triangles I and II outlined in the middle diagram), so the shorter base is \(\frac{d}{3}\). The trapezoid's area is \(\frac{h}{2} (b_1+b_2) = \frac{d}{6} (\frac{9}{4} + \frac{d}{3}) = \frac{5d^2}{48} = 40\).

Method II: Two of the shaded triangle's vertices are midpoints of sides of the square, so the smaller unshaded region is \(1/8\) of the square, and the other unshaded regions are each \(1/4\) of the square. The shaded triangle's area is \((3/8) \times 192 = 72\). Notice that \(\triangle I\) is similar to \(\triangle II\). Since \(M\) is a midpoint, the parallel sides of \(\triangle I\) and \(II\) are in the ratio \(x:2x = 1:2\), as are all corresponding sides. Thus, the smaller shaded triangle and the larger shaded triangle are similar, with ratio of similitude \(2k:3k = 2:3\). Since the area ratio of similar triangles is the square of their ratio of similitude, the ratio of the areas of the shaded triangles is \((2/3)^2 = 4:9\). The smaller shaded triangle's area = \((4/9) \times 72 = 32\). The trapezoid's area is what remains: \(72 - 32 = 40\).