HIGH SCHOOL MATHEMATICS CONTESTS

Math League Press, P.O. Box 17, Tenafly, New Jersey 07670-0017

All official participants must take this contest at the same time.

Contest Number 5 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **February 12, 2019**

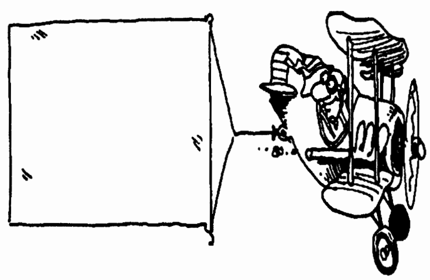
Name ________________________ **Teacher** ____________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes NEXT CONTEST: MAR. 19, 2019 *Answer Column*

5-1. If N is a 3-digit integer, and given that the result of reversing the digits of N is the number M, what is the maximum value of $N-M$? [Note: Integers written with 1 or more leading zeros can also be written with every leading zero removed.] — 5-1.

5-2. If a and b are integers whose product is 5, what is the least possible value of a^b? — 5-2.

5-3. Al's biplane can fly two different types of square advertising signs. Both signs have integer side-lengths. Their areas are $n+1$ and $2n+1$. What is the least positive integer n for which each sign's area is the square of an integer? — 5-3.

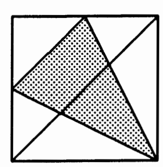


5-4. In a group of 100, each a Borg or a Corg, every Borg has 2 Corg friends and no Corg has more than 1 Borg friend. If exactly 13 Corgs have no Borg friends, then how many Corgs are in the group? — 5-4.

5-5. If 2019! is the product of the first 2019 positive integers, for which positive integer k is $x = 2019!$ the only value of $x > 0$ that satisfies — 5-5.

$$\frac{1}{\log_4 x} + \frac{1}{\log_9 x} + \frac{1}{\log_{16} x} + \ldots + \frac{1}{\log_{(2019^2)} x} = k?$$

5-6. Midpoints of two sides of a square are vertices of the shaded triangle shown. Drawing the diagonal of the square pictured splits this triangle into two parts, one of which is a trapezoid. If the area of the square is 192, what is the area of the trapezoid? — 5-6.



Contest # 5 — *Answers & Solutions* — 2/12/19

Problem 5-1

Since making N's hundreds' digit a 9 and its units' digit a 0 will maximize the difference, no matter what the middle digit is, $N-M = 9\underline{?}0 - 0\underline{?}9 = \boxed{891}$.

Problem 5-2

Since $ab = 5$, and a and b are integers, (a,b) must be one of $(1,5)$, $(5,1)$, $(-1,-5)$, or $(-5,-1)$. The least value of a^b is $(-1)^{-5} = \boxed{-1}$.

Problem 5-3

By trial and error, take each positive perfect square and subtract 1 to find every positive integer n for which $n+1$ is a perfect square. Find the least of those integers n for which $2n+1$ is the square of an integer. Let's try 1. That fails because $1-1 = 0$ is not positive. Try 4: $4-1 = 3$, but $2(3)+1 = 7$, which is not a perfect square. We try 9, then 16, and when we finally try 25, we get $25-1 = 24$, and $2(24)+1 = 49$. Both $n+1$ and $2n+1$ are perfect squares when $n = \boxed{24}$.

Problem 5-4

If B is the number of Borgs, and C is the number of Corgs, then the number of Corgs is $2B+13$. Therefore, $B+(2B+13) = 100$, $B = 29$, and $C = \boxed{71}$.

Problem 5-5

Whenever $a > 1$ and $b > 1$, $\log_a b = \dfrac{1}{\log_b a}$. We can write $\log_x 4 + \log_x 9 + \log_x 16 + \ldots + \log_x 2019^2 = k$, or $\log_x(2^2 \times 3^2 \times 4^2 \times \ldots \times 2019^2) = k$. It follows that $x^k = (2^2 \times 3^2 \times 4^2 \times \ldots \times 2019^2) = (2019!)^2$, so $k = \boxed{2}$.

Problem 5-6

Method I: The square's side is $8\sqrt{3}$, so its diagonal is $8\sqrt{6} = d$. The trapezoid's height is $\frac{d}{4}$. Its longer base is $\frac{d}{2}$. The shaded $\triangle$'s legs trisect the diagonal (to prove this, use similar triangles I and II outlined in the middle diagram), so the shorter base is $\frac{d}{3}$. The trapezoid's area is $\frac{h}{2}(b_1+b_2) = \frac{d}{8}(\frac{d}{2}+\frac{d}{3}) = \frac{5d^2}{48} = \boxed{40}$.

Method II: Two of the shaded triangle's vertices are midpoints of sides of the square, so the smaller unshaded region is 1/8 of the square, and the other unshaded regions are each 1/4 of the square. The shaded triangle's area is $(3/8) \times 192 = 72$. Notice that $\triangle$I is similar to $\triangle$II. Since M is a midpoint, the parallel sides of $\triangle$s I and II are in the ratio $x{:}2x = 1{:}2$, as are all corresponding sides. Thus, the smaller shaded triangle and the larger shaded triangle are similar, with ratio of similitude $2k{:}3k = 2{:}3$. Since the area ratio of similar triangles is the square of their ratio of similitude, the ratio of the areas of the shaded triangles is $(2/3)^2 = 4{:}9$. The smaller shaded triangle's area $= (4/9) \times 72 = 32$. The trapezoid's area is what remains: $72-32 = 40$.